Deformation Quantization

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based on Chapter 5 of Deformation Quantization and Index Theory by B. Fedosov

Let (M, ω) be a symplectic manifold, and let $Z = C^\infty(M)[\![\hbar]\!]$ be the linear space of formal power series

$$a = \sum_{k=0}^{\infty} \hbar^k a_k, \quad \text{with } a_k \in C^\infty(M).$$

Definition 1. Deformation quantization of $C^\infty(M)$ refers to an associative product $\star$ on Z, called a **star product**, satisfying

1. (*formal deformation*) $a \star b \mod \hbar = ab$ for all $a, b \in C^\infty(M)$.
2. (*locality*) for any $a, b \in Z$, we have $a \star b = \sum_{k=0}^{\infty} \hbar^k c_k$, where c_k depends on $\partial^\alpha a_i \partial^\beta b_j$ with $i + j + |\alpha| + |\beta| \leq k$.
3. (*correspondence principle*) for all $a, b \in Z$, we have

   $$[a, b] = a \star b - b \star a = -i\hbar\{a_0, b_0\} + \mathcal{O}(\hbar^2),$$

   where $\{\cdot, \cdot\}$ denotes the Poisson associated to ω.

Remark 2. Note that deformation quantization differs from Weyl quantization by the fact that the Planck constant $\hbar$ is no longer a positive number, but a formal parameter.

The Formal Weyl Algebras Bundle

Definition 3. The **formal Weyl algebra bundle** is the bundle $W = \widehat{\text{Sym}}(T^*M \otimes \mathbb{C})[\![\hbar]\!]$. Locally, its sections are of the form

$$a = \sum_{k, |\alpha| \geq 0} \hbar^k a_{k,\alpha} y^\alpha,$$

where $y^\alpha = (y^1)^{\alpha_1} \cdots (y^{2n})^{\alpha_{2n}}$, with y^i a basis for T^*M, and $a_{k,\alpha}$ complex-valued functions on M.

Definition 4. The **Weyl product** of two sections $a, b \in \Gamma(W)$ is given (fiberwise) by

$$\begin{aligned} a \circ b &= \exp\left(-\frac{i\hbar}{2}\omega^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right) a(y)b(z)\bigg|_{z=y} \\ &= \sum_{k=0}^{\infty}\left(-\frac{i\hbar}{2}\right)^k \frac{1}{k!}\omega^{i_1 j_1}\cdots\omega^{i_k j_k}\frac{\partial^k a}{\partial y^{i_1}\cdots\partial y^{i_k}}\frac{\partial^k b}{\partial y^{j_1}\cdots\partial y^{j_k}}. \end{aligned}$$

Lemma 5. *The center of* $\Gamma(W)$ *with respect to the Weyl product is* Z.

Proof. Take any a in the center of $\Gamma(W)$. If we take $b = y^k$ for some k, then

$$a \circ b = a y^k - \frac{i\hbar}{2}\omega^{ik}\frac{\partial a}{\partial y^i} \qquad \text{and} \qquad b \circ a = a y^k - \frac{i\hbar}{2}\omega^{kj}\frac{\partial a}{\partial y^j},$$

so

$$0 = [a, b] = -i\hbar\omega^{ik}\frac{\partial a}{\partial y^i}.$$

Varying over k, we find that $\frac{\partial a}{\partial y^i} = 0$ for all i, so $a \in Z$. Conversely, it is easy to see that Z lies in the center of W. □

We grade the bundle W by setting $\deg y^i = 1$ and $\deg \hbar = 2$. This yields a filtration

$$\Gamma(W) \supset \Gamma(W_1) \supset \Gamma(W_2) \supset \cdots$$

Similarly, the bundles of differential forms $W \otimes \Lambda^q$ are graded, where the degree of any pure q-form is zero. The Weyl product can be extended to $W \otimes \Lambda$ using the wedge product $\wedge$, where the y^i and dx^i commute. The commutator of forms $a \in \Gamma(W \otimes \Lambda^{q_1})$ and $b \in \Gamma(W \otimes \Lambda^{q_2})$ is

$$[a, b] = a \circ b - (-1)^{q_1 q_2} b \circ a.$$

Similar to Lemma 5, the center of $\Gamma(W \otimes \Lambda)$ with respect to the Weyl product is $Z \otimes \Lambda$.

Notation 6. For any $a \in \Gamma(W \otimes \Lambda)$, we write $a_0 = a|_{y=0}$ and $a_{00} = a|_{y=0,dx=0}$. Furthermore, for any $a \in \Gamma(W)$, we write $\sigma(a)$ for $a_0 = a|_{y=0}$.

Definition 7. Define operations δ and δ^* on $\Gamma(W \otimes \Lambda)$ by

$$\delta : \Gamma(W_p \otimes \Lambda^q) \to \Gamma(W_{p-1} \otimes \Lambda^{q+1}), \quad a \mapsto dx^k \wedge \frac{\partial a}{\partial y^k},$$
$$\delta^* : \Gamma(W_p \otimes \Lambda^q) \to \Gamma(W_{p+1} \otimes \Lambda^{q-1}), \quad a \mapsto y^k \iota_{\partial_{x^k}} a.$$

In particular, δ lowers the degree by one, while δ^* raises the degree by one.

Lemma 8. *The operations δ and δ^* do not depend on the choice of local coordinates, and satisfy*

(i) $\delta^2 = (\delta^*)^2 = 0,$

(ii) $(\delta\delta^* + \delta^*\delta)(a) = (p+q)a$ *for a monomial* $a = y^{i_1} \cdots y^{i_p} dx^{j_1} \wedge \cdots \wedge dx^{j_q}$.

(iii) $\delta(a \circ b) = (\delta a) \circ b + (-1)^{q_1} a \circ (\delta b)$ *for* $a \in \Gamma(W \otimes \Lambda^{q_1})$ *and* $b \in \Gamma(W \otimes \Lambda^{q_2})$.

(iv) $\delta a = -\frac{i}{\hbar}[\omega_{ij} y^i dx^j, a]$.

Proof. Straightforward. □

Definition 9. Let $a \in \Gamma(W \otimes \Lambda)$, and write a_{pq} for (p, q)-homogeneous part. Then define

$$\delta^{-1} a_{pq} = \begin{cases} \frac{1}{p+q}\delta^* a_{pq} & \text{if } p + q > 0, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, using Lemma 8(*ii*), any $a \in \Gamma(W \otimes \Lambda)$ has a *Hodge–De Rham decomposition*

$$a = a_{00} + \delta\delta^{-1}a + \delta^{-1}\delta a. \tag{1}$$

Recall that there exists a symplectic connection ∇ on M. Tensorially, there is an induced connection on $W \otimes \Lambda$, also denoted by ∇.

Lemma 10.

(i) $\nabla(a \circ b) = \nabla a \circ b + (-1)^{q_1} a \circ \nabla b$ *for* $a \in \Gamma(W \otimes \Lambda^{q_1})$.

(ii) $\nabla(\eta \wedge a) = d\eta \wedge a + (-1)^q \eta \wedge \nabla a$ *for* $\eta \in \Gamma(\Lambda^q)$.

Proof. Follows from the definition of the Weyl product $\circ$ and the fact that ∇ preserves ω. □

Let us work in Darboux local coordinates, with Γ_{ij}^k the Christoffel symbols. Recall that for a symplectic connection the numbers $\Gamma_{ijk} = \omega_{i\ell}\Gamma_{jk}^{\ell}$ are completely symmetric in ijk. Although it is cumbersome to write out, it is straightforward to find that

$$\nabla a = da + \frac{i}{\hbar}\left[\frac{1}{2}\Gamma_{ijk} y^i y^j dx^k, a\right],$$

and we write $\Gamma = \frac{1}{2}\Gamma_{ijk} y^i y^j dx^k$ for the local 1-form with values in W.

Now, we want to consider more general (symplectic) connections. Consider connections of the form

$$Da = \nabla a + \frac{i}{\hbar}[\gamma, a] = da + \frac{i}{\hbar}[\Gamma + \gamma, a],$$

where $\gamma \in \Gamma(W \otimes \Lambda^1)$, a global 1-form. Note that γ is determined by D only up to a central one-form, since it appears in a commutator. To enforce uniqueness we impose the *Weyl normalization condition*, requiring $\gamma_0 = \gamma|_{y=0} = 0$ (like a gauge condition).

Lemma 11. *Let* ∇ *be a symplectic connection on* M. *Then*

$$\nabla\delta a + \delta\nabla a = 0$$

and

$$\nabla^2 a = \frac{i}{\hbar}[R, a]$$

where $R = \frac{1}{4}R_{ijk\ell} y^i y^j dx^k \wedge dx^\ell$, *with* $R_{ijk\ell}$ *is the curvature tensor of* ∇.

Proof. Follows from the expression of ∇ and δ as above. Note that the latter equation is a compact form of the Ricci identity. □

Definition 12. Let D be a connection on W of the form $D = \nabla + \frac{i}{\hbar}[\gamma, \cdot]$ with $\gamma_0 = 0$. Then the **curvature** of D is defined as

$$\Omega = R + \nabla\gamma + \frac{i}{\hbar}\gamma^2.$$

Lemma 13. *We have*

(i) (Bianchi identity) $D\Omega = \nabla\Omega + \frac{i}{\hbar}[\gamma, \Omega] = 0$,

(ii) (Ricci identity) $D^2 a = \frac{i}{\hbar}[\Omega, a]$.

Proof. By definition of D and Ω, we have

$$D\Omega = \nabla R + \nabla^2\gamma + \frac{i}{\hbar}[\nabla\gamma, \gamma] + \frac{i}{\hbar}[\gamma, R] + \frac{i}{\hbar}[\gamma, \nabla\gamma] + \left(\frac{i}{\hbar}\right)^2 [\gamma, \gamma^2].$$

By the Bianchi identity for ∇, we have $\nabla R = 0$. Furthermore, obviously $[\gamma, \gamma^2] = 0$, and $\nabla^2\gamma = \frac{i}{\hbar}[R, \gamma]$ as seen earlier. Therefore, $D\Omega = 0$. Part *(ii)* is straightforward. □

Abelian Connections and Quantization

Definition 14. A connection D of W is **abelian** if

$$D^2 a = \frac{i}{\hbar}[\Omega, a] = 0$$

for all $a \in \Gamma(W \otimes \Lambda)$, that is, if the curvature of the connection is a central form.

We will show there exists an abelian connection of the form

$$D = \nabla - \delta + \frac{i}{\hbar}[r, \cdot] = \nabla + \frac{i}{\hbar}[\omega_{ij} y^i dx^j + r, \cdot],$$

where ∇ is a fixed symplectic connection, and $r \in \Gamma(W_3 \otimes \Lambda^1)$ a globally defined one-form, with Weyl normalization $r_0 = 0$. Computing the curvature of D gives

$$\Omega = -\frac{1}{2}\omega_{ij} dx^i \wedge dx^j + R - \delta r + \nabla r + \frac{i}{\hbar} r^2.$$

It suffices to find an r satisfying

$$\delta r = R + \nabla r + \frac{i}{\hbar} r^2,$$

so that $\Omega = -\omega$ is indeed central.

Theorem 15. *The above equation has a unique solution r such that $\deg r \geq 2$ and $\delta^{-1} r = 0$.*

Proof. From (1) follows that any such r has $r = \delta^{-1}\delta r$, as $r_{00} = 0$ and $\delta\delta^{-1} r = 0$. Applying δ^{-1} yields

$$r = \delta^{-1} R + \delta^{-1}\left(\nabla r + \frac{i}{\hbar} r^2\right).$$

Since ∇ preserves the filtration on $W \otimes \Lambda$, and δ^{-1} raises the degree by 1, one obtains a unique solution by the iteration method. Conversely, one can show that this solution yields an abelian connection (again using the iteration method). □

Remark 16. Explicitly, the iterating method yields

$$r = \frac{1}{8} R_{ijk\ell} y^i y^j y^k dx^\ell + \frac{1}{20} \nabla_m R_{ijk\ell} y^i y^j y^k y^m dx^\ell + \cdots$$

Definition 17. Let D be an abelian connection on W. We define $W_D \subset W$ to be the subbundle of flat sections with respect to D, that is, $Da = 0$. Note that $\Gamma(W_D)$ is a subalgebra of $\Gamma(W)$ with respect to the Weyl product because of Lemma 10.

Theorem 18. *For any $a_0 \in Z$, there exists a unique section $a \in \Gamma(W_D)$ such that $\sigma(a) = a_0$.*

Proof. Rewrite the equation $Da = 0$ as

$$\delta a = (D + \delta) a,$$

and note that $D+\delta = \nabla + \frac{i}{\hbar}[r,\cdot]$ does not lower degree since $\deg r \geq 2$. Applying δ^{-1}, we find using the (1) that

$$a = a_0 + \delta^{-1}\delta a = a_0 + \delta^{-1}(D+\delta)a, \qquad (*)$$

where we used $\delta\delta^{-1}a = 0$ as $a \in \Gamma(W)$. Since δ^{-1} raises degree, we can solve this equation (uniquely) via iterations. Conversely, if a is a solution of $(*)$, then $\sigma(a) = a_0$ since $\sigma \circ \delta^{-1} = 0$. Now,

$$\delta^{-1}Da = \delta^{-1}(D+\delta)a - \delta^{-1}\delta a = a - a_0 - \delta^{-1}\delta a = \delta\delta^{-1}a = 0.$$

Since D is abelian, we have $D(Da) = 0$, or $\delta Da = (D+\delta)Da$, and applying δ^{-1} gives

$$Da = \delta^{-1}(D+\delta)Da.$$

Solve by iterations to get $Da = 0$. □

Remark 19. By iterations, we can construct the section $a \in \Gamma(W_D)$ from its symbol $a_0 = \sigma(a)$,

$$a = a_0 + \partial_i a_0 y^i + \frac{1}{2}\partial_i\partial_j a_0 y^i y^j + \frac{1}{6}\partial_i\partial_j\partial_k a_0 y^i y^j y^k - \frac{1}{24}R_{ijk\ell}\omega^{\ell m}\partial_m a_0 y^i y^j y^k + \cdots$$

If the curvature tensor R is zero, we have

$$a = \sum_{k=0}^{\infty} \frac{1}{k!}\partial_{i_1}\cdots\partial_{i_k} a_0 y^{i_1}\cdots y^{i_k}.$$

Definition 20. The bijection between $\Gamma(W_D)$ and $Z = C^\infty(M)[\![\hbar]\!]$ allows to define a *star product* on Z, given by

$$a \star b = \sigma(Q(a) \circ Q(b)),$$

where $Q : Z \to W_D$, called the *quantization procedure*, is the inverse to σ. One can check that this star product satisfies the properties of Definition 1. The subalgebra $\Gamma(W_D)$ is called the *quantum algebra.*

Example 21. Let $M = \mathbb{R}^{2n}$ with $\omega = \frac{1}{2}\omega_{ij}dx^i \wedge dx^j$ a constant symplectic form on M. The connection

$$D^0 a = da + \frac{i}{\hbar}[\omega_{ij}y^i dx^j, a] \qquad \text{or} \qquad D^0 = d - \delta,$$

is abelian with curvature

$$\Omega = -\omega.$$

Now the corresponding quantum algebra is given by

$$\Gamma(W_{D^0}) = \left\{ a \in \Gamma(W) : \frac{\partial a}{\partial x^i} - \frac{\partial a}{\partial y^i} = 0 \right\}.$$

That is, any $a \in \Gamma(W_{D^0})$ is of the form

$$a = \sum_{|\alpha|\geq 0} \frac{1}{|\alpha|!}\partial_\alpha b y^\alpha,$$

for some $b \in Z = C^\infty(\mathbb{R}^{2n})[\![\hbar]\!]$. Note that the star product now corresponds to the Weyl product.

Remark 22. Later it will be shown that any W_D is locally isomorphic to $W_{D^0}(\mathbb{R}^{2n})$.

Theorem 23. *The cohomology groups of*

$$\cdots \to \Gamma(W \otimes \Lambda^p) \xrightarrow{D} \Gamma(W \otimes \Lambda^{p+1}) \to \cdots$$

are trivial for $p > 0$.

Proof. We can extend the quantization procedure to an isomorphism $Q : \Gamma(W \otimes \Lambda^p) \xrightarrow{\sim} \Gamma(W \otimes \Lambda^p)$ via

$$Qa = a + \delta^{-1}(D + \delta)Qa.$$

Indeed, by the iterating method there is a unique solution, and the inverse is given by

$$Q^{-1}a = a - \delta^{-1}(D + \delta)a.$$

One can show that

$$Q^{-1}D + \delta Q^{-1} = 0,$$

by substituting for Q^{-1}, and using (1). Then it follows that $D = -Q\delta Q^{-1}$, so we can replace the complex with $-\delta$, and then the result follows from the Hodge–De Rham decomposition. Namely, for any $a \in \Gamma(W \otimes \Lambda^p)$, write $a = a_{00} + \delta\delta^{-1}a + \delta^{-1}\delta a$. If $\delta a = 0$, that is,

$$\delta a = \delta a_{00} + \delta\delta^{-1}\delta a = \delta a_{00} + (\delta a - \delta^{-1}\delta^2 a) = \delta a_{00} + \delta a = 0,$$

then $a = a_{00} + \delta\delta^{-1}a + \delta^{-1}\delta a = a_{00} - \delta^{-1}\delta a_{00} + \delta\delta^{-1}a = \delta(\delta^{-1}a_{00} + \delta^{-1}a)$ lies in the image of δ, so the sequence is exact for $p > 0$. Note that we used that $p > 0$ in the line where $\delta\delta^{-1}a + \delta^{-1}\delta a = a$. □

Corollary 24. *Any equation* $Da = b$ *with* $b \in \Gamma(W \otimes \Lambda^p)$ *and* $p > 0$ *has a solution if and only if* $Db = 0$. *The solution may be taken in the form*

$$a = D^{-1}b = -Q\delta^{-1}Q^{-1}b.$$

Generalizations

Note that in the above, the symplectic form ω pops up in two places: in the Weyl multiplication rule, and as the curvature of the abelian connection D. In this section we will make them distinct. This is convenient when when we have to vary symplectic structures: we may fix the Weyl multiplication and vary the curvature.

Let L be a symplectic vector bundle over M of dimension $2n$ with a fixed symplectic structure ω and symplectic connection ∇^L. We assume that L is isomorphic to TM, but not canonically. Denote by

$$\theta : TM \to L$$

a bundle isomorphism, and by

$$\delta : L^* \to T^*M$$

a dual isomorphism. Introducing a local symplectic frame $(e_1, \ldots, e_{2n})$ for L yields a dual frame $(e^1, \ldots, e^{2n})$ for L^*, and a frame $\theta^1 = \delta(e^1), \ldots, \theta^{2n} = \delta(e^{2n})$ for T^*M, with corresponding vector fields $X_1, \ldots, X_{2n}$ giving a dual frame for TM. The form ω on L can be transported to TM giving a non-degenerate 2-form on M

$$\Omega_0 = -\frac{1}{2}\omega_{ij}\theta^i \wedge \theta^j,$$

but note that it need not be closed. We will use θ to vary the symplectic structure on TM.

Lemma 25. *Let $\Omega(t)$ be a family of non-degenerate 2-forms on M with $\Omega(0) = \Omega_0 = -\frac{1}{2}\omega_{ij}\theta^i \wedge \theta^j$. Then there exists a family $\theta(t)$ of isomorphisms such that $\Omega(t) = -\frac{1}{2}\omega_{ij}\theta(t)^i \wedge \theta(t)^j$.*

Proof. Omitted. See [1, Lemma 5.3.1]. □

Analogous to the previous section, we make some definitions.

Definition 26. Let $\mathcal{E}$ be a complex vector bundle over M with connection $\nabla^{\mathcal{E}}$, and let $\mathcal{A} = \mathrm{Hom}(\mathcal{E}, \mathcal{E})$ (the *coefficient bundle*).

- The *formal Weyl bundle with coefficients in* $\mathcal{A}$ is the bundle

$$W(L, \mathcal{A}) = \widehat{\mathrm{Sym}}(L^*)[\![\hbar]\!] \otimes \mathcal{A}.$$

- Using the same rule as in Definition 3, we can define Weyl multiplication on $W(L, \mathcal{A})$, but now the coefficients are taken in $\mathcal{A}$, which means the multiplication may be non-commutative.

- The connections ∇^L and $\nabla^{\mathcal{E}}$ induce a connection ∇ on $W(L, \mathcal{A})$.

- In a local symplectic frame of L, we can write

$$\nabla a = da + \frac{i}{\hbar}\left[\frac{1}{2}\Gamma_{ij}y^i y^j, a\right] + [\Gamma_{\mathcal{E}}, a] \quad \text{and} \quad \nabla^2 a = \frac{i}{\hbar}\left[\frac{1}{2}R_{ij}y^i y^j, a\right] + [R_{\mathcal{E}}, a],$$

so we define the curvature of ∇ to be

$$R = \frac{1}{2}R_{ij}y^i y^j - i\hbar R_{\mathcal{E}} \in \Gamma(W(L, \mathcal{A}) \otimes \Lambda^2).$$

- Consider more general connections on $W(L, \mathcal{A})$ of the form

$$D = \nabla + \frac{i}{\hbar}[\gamma, a],$$

for some globally defined $\gamma \in \Gamma(W(L, \mathcal{A}) \otimes \Lambda^1)$. (Note that there are no unique ∇ and γ representing D, although we can always choose an arbitrary symplectic connection ∇, and then γ is well-defined up to some scalar 1-form $\Delta\gamma \in \Gamma(\Lambda^1)[\![\hbar]\!]$.) The curvature of D (with respect to ∇ and γ) is defined by

$$\Omega = \nabla\gamma + \frac{i}{\hbar}\gamma^2 + R \in \Gamma(W(L, \mathcal{A}) \otimes \Lambda^2).$$

- Define operators

$$\delta : \Gamma(W(L, \mathcal{A})_p \otimes \Lambda^q) \to \Gamma(W(L, \mathcal{A})_{p-1} \otimes \Lambda^{q+1}), \quad a \mapsto \theta^k \wedge \frac{\partial a}{\partial y^k},$$
$$\delta^* : \Gamma(W(L, \mathcal{A})_p \otimes \Lambda^q) \to \Gamma(W(L, \mathcal{A})_{p+1} \otimes \Lambda^{q-1}), \quad a \mapsto y^k \iota_{X_k} a.$$

In particular, note that δ agrees with $L^* \to T^*M$ on linear forms.

- The construction of δ^{-1}, the Bianchi identity and Ricci identity (Lemma 13), the Hodge–De Rham decomposition (1), all remain valid.

Theorem 27. *Let $\Omega = \Omega_0 + \hbar\Omega_1 + \hbar^2\Omega_2 + \cdots$ be a closed 2-form, and $\theta : TM \to L$ a bundle isomorphism such that $\Omega_0 = -\frac{1}{2}\omega_{ij}\theta^i \wedge \theta^j$. Then for any section $\mu \in \Gamma(W(L, \mathcal{A}))$ with $\deg(\mu) \geq 3$ and $\mu|_{y=0} = 0$ there exists a unique section $r \in \Gamma(W(L, \mathcal{A}) \otimes \Lambda^1)$ with $\deg(r) \geq 2$ such that $\delta^{-1}r = \mu$, and the corresponding connection $D = \nabla - \delta + \frac{i}{\hbar}[r, \cdot]$ is abelian with curvature Ω.*

Proof. Omitted. See [1, Theorem 5.3.3]. □

Remark 28. The construction of D as in the theorem depends smoothly on the parameters. That is, if $\Omega(t)$ is a family of closed 2-forms with non-degenerate leading term $\Omega_0(t)$, and a family $\mu(t)$ with $\deg \mu(t) \geq 3$ and $\mu(t)|_{y=0} = 0$, there exists a family $r(t)$ satisfying the requirements.

Having constructed the abelian connection D, we define a quantum algebra with twisted coefficients $W_D(L, \mathcal{A})$ in the same way as before. Theorems 18 and 23 and Corollary 24 remain valid for the bundle $W(L, \mathcal{A})$. In particular, may define a quantization procedure

$$\Gamma(\mathcal{A})[\![\hbar]\!] \underset{\sigma}{\overset{Q}{\rightleftarrows}} \Gamma(W_D(L, \mathcal{A})).$$

The Heisenberg Equation

Consider the *Heisenberg equation* in $W_D = W_D(L, \mathcal{A})$,

$$\frac{da}{dt} + \frac{i}{\hbar}[H(t), a] = 0, \tag{2}$$

with $H(t) \in \Gamma(W_D)$ a given flat section, and $a(t) \in \Gamma(W_D)$ an unknown flat section. If $H(t)$ and $a(t)$ are obtained via quantization, coming from symbols $H_0(t)$ and $a_0(t)$, then the leading term of the equation reads

$$\frac{d}{dt}a_0(t) + \{H_0, a_0\} = 0,$$

which corresponds to the Louiville equation in classical mechanics. That is, the Heisenberg equation can be seen as the *quantum analogue* of the Liouville equation.

Consider a family of abelian connections on $W(L, \mathcal{A})$,

$$D_t = \nabla + \frac{i}{\hbar}[\gamma_t, \cdot] = \nabla - \delta_t + \frac{i}{\hbar}[r(t), \cdot],$$

where $\gamma_t = \omega_{ij}y^i\theta(t)^j + r(t)$ with $\deg(r(t)) \geq 2$, and $\theta(t) : TM \to L$ is a family of bundle isomorphisms. Furthermore, let $H(t)$ be a section of $W(L, \mathcal{A})$, called the **Hamiltonian**, satisfying

(1) $\lambda := D_tH(t) - \dot{\gamma}(t)$ lies in $\Lambda^1[\![\hbar]\!]$,

(2) there exists a vector field X_t such that $\deg\left(\iota_{X_t}\Gamma(t) + H(t)\right) \geq 2$.

Now consider the equation

$$\frac{da}{dt} + \left(\iota_{X_t}D_t + D_t\iota_{X_t}\right)a + \frac{i}{\hbar}[H(t), a] = 0. \tag{3}$$

Remark 29. When D is time-independent and $a \in \Gamma(W_D)$, the above equation reduces to (2). Namely, in this case $\iota_{X_t} a = 0$ and $Da = 0$, so $(\iota_{X_t} D_t + D_t \iota_{X_t})\, a = 0$. Furthermore, λ is closed since $d\lambda = D\lambda = D^2 H = 0$, as D is abelian, so locally we can write $\lambda = -dH_0(t)$ for some scalar function $H_0(t)$. Since $H_0(t)$ is central, we can replace $H(t)$ with $H(t) + H_0(t)$, which is flat as $D(H(t) + H_0(t)) = 0$ by the first property of the Hamiltonian.

Definition 30. Let $W^+ \supset W$ be the bundle whose sections are of the form

$$a = \sum_{2k+|\alpha|\geq 0} \hbar^k a_{k,\alpha} y^\alpha,$$

where k is allowed to be negative, as long as the total degree $2k + |\alpha|$ is non-negative.

Remark 31. Note that the fibers W_x^+ are still algebras with respect to the Weyl multiplication, and the connections ∇ and D are well-defined on W^+.

Lemma 32. *Let $a \in \Gamma(W^+)$ with $Da = 0$, then $a \in \Gamma(W_D)$. That is, a does not contain negative powers of $\hbar$.*

Proof. Note that $\sigma(a)$ must only have non-negative powers of $\hbar$, and thus $\sigma(a) \in Z$. By Theorem 18, a flat section is determined by $\sigma(a)$, so it follows that $a \in \Gamma(W_D)$. □

Assume that the vector field X_t defines a flow $f_t : M \to M$ for $t \in [0, 1]$. (Generally this is only true for small t and $x \in M$ ranging over a compact set.)

Theorem 33. *For any initial $a(0) \in \Gamma(W \otimes \Lambda)$, equation (3) has a unique solution $a(t) \in \Gamma(W \otimes \Lambda)$. Moreover, if $a(0) \in \Gamma(W_{D_0})$, then $a(t) \in \Gamma(W_{D_t})$.*

Proof. Substituting $D_t = \nabla + \frac{i}{\hbar}[\gamma_t, \cdot]$, we can rewrite (3) as

$$\frac{da}{dt} + (\iota_{X_t} \nabla + \nabla \iota_{X_t})\, a + \frac{i}{\hbar}[H(t) + \iota_{X_t}\gamma_t, a] = 0.$$

By the second property of the Hamiltonian, we know $\deg\left(H(t) + \iota_{X_t}\gamma_t\right) \geq 2$, so we write

$$H(t) + \iota_{X_t}\gamma_t = H_2(t) + H_3(t) = \frac{1}{2} H_{ij}(t) y^i y^j + \hbar\mathcal{H}(t) + H_3(t),$$

where $\mathcal{H}(t)$ is a section of $\mathcal{A}$ and $\deg H_3(t) \geq 3$. We define a pullback $f_t^* : W \otimes \Lambda \to W \otimes \Lambda$ as follows. On differential forms it is the usual pullback, and on sections $a \in \Gamma(W)$ we set

$$(f_t^* a)(x, y) = v_t^{-1} a(f_t(x), \sigma_t(y)) v_t,$$

where $\sigma_t : L_x \to L_{f_t(x)}$ is a linear symplectic lifting of f_t, and $v_t : \mathcal{E} \to \mathcal{E}_{f_t(x)}$ an isomorphism of bundles, lifting f_t. Now we will see how these lifts are obtained.

Lemma 34. *There exist such liftings σ_t and v_t such that for any $a \in \Gamma(W \otimes \Lambda)$,*

$$\frac{d}{dt}\left(f_t^* a\right) = f_t^* \left((\iota_{X_t} \nabla + \nabla \iota_{X_t}) a + \frac{i}{\hbar}[H_2(t), a] \right).$$

Proof. For scalar differential forms, the above equation follows from Cartan's formula, so it suffices to prove the equation for $a \in \Gamma(W)$. See [1, Lemma 5.4.4]. $\square$

Now, for any solution $a(t)$ of (3), we have that $b(t) = f_t^* a(t)$ satisfies

$$\begin{aligned}
&\frac{d}{dt} b(t) + \frac{i}{\hbar}[f_t^* H_3, b] \\
&= f_t^* \left(\frac{da}{dt} + (\iota_{X_t} \nabla + \nabla \iota_{X_t})\, a + \frac{i}{\hbar}[H_2(t), a] \right) + f_t^* \left(\frac{i}{\hbar}[H_3(t), a] \right) \\
&= 0,
\end{aligned}$$

and conversely, any such $b(t)$ gives $a(t) = (f_t^*)^{-1} b(t)$ a solution of (3). Hence, it suffices to solve for $b(t)$,

$$b(t) = b(0) - \frac{i}{\hbar} \int_0^t [f_\tau^* H_3(\tau), b(\tau)] d\tau,$$

which can be done via iterations. Indeed these iterations converge as $\deg(f_t^* H_3(t)) \geq 3$.

Remark 35. The solution $b(t)$ can be expressed in a shortened form as

$$b(t) = U^{-1}(t) \circ b(0) \circ U(t),$$

where

$$U(t) = \mathrm{Pexp} \left(\frac{i}{\hbar} \int_0^t f_t^* H_3(t) dt \right)$$

is defined by a *path-ordered exponential*, that is,

$$U(t) = 1 + \frac{i}{\hbar} \int_0^t (f_\tau^* H_3(\tau) \circ U(\tau)) d\tau.$$

Indeed, such a solution for $U(t)$ exists as $\deg\left(\frac{i}{\hbar} f_\tau^* H_3(\tau)\right) \geq 1$.

It remains to prove the last assertion of the theorem, for which we need the following lemma.

Lemma 36. *For any solution $a(t)$, also $D_t a(t)$ is a solution.*

Proof. We have

$$\frac{d}{dt}(D_t a) = \frac{d}{dt} \left(\nabla a + \frac{i}{\hbar}[\gamma_t, a] \right) = \nabla \dot{a} + \frac{i}{\hbar}[\gamma_t, \dot{a}] + \frac{i}{\hbar}[\dot{\gamma}_t, a] = D_t \frac{da}{dt} + \frac{i}{\hbar}[\dot{\gamma}_t, a].$$

Since $a(t)$ is a solution, we can substitute for $\frac{da}{dt}$ to obtain

$$\begin{aligned}
\frac{d}{dt}(D_t a) &= -D_t \iota_{X_t} D_t a - \frac{i}{\hbar}[D_t H(t), a] - \frac{i}{\hbar}[H(t), D_t a] + \frac{i}{\hbar}[\dot{\gamma}_t, a] \\
&= -\left(\iota_{X_t} D_t + D_t \iota_{X_t} \right) D_t a - \frac{i}{\hbar}[H(t), D_t a],
\end{aligned}$$

using that $D_t^2 = 0$ and the fact that $D_t H - \dot{\gamma}_t$ is central by the first property of the Hamiltonian. $\square$

Finally, if $a(t)$ is a solution, then so is $D_t a(t)$ by the above lemma, so whenever $D_0 a(0) = 0$ it follows from the uniqueness of the solution that $D_t a(t) = 0$ for all t. $\square$

Corollary 37. *Let D be time-independent, and let $H(t) \in \Gamma(W_D)$ be a flat section with scalar leading term $H_0(t)$. Then for any $a(0) \in \Gamma(W_D)$ there exists a unique solution $a(t) \in \Gamma(W_D)$, and the map $A(t) : a(0) \mapsto a(t)$ is an automorphism of W_D.*

Proof. The difference $H(t) - H_0(t)$ satisfies the properties of the Hamiltonian, and so the result follows from the above theorem. □

References

[1] B. Fedosov, Deformation Quantization and Index Theory